

The Lagrangian for the Universe

Nikodem Popławski

*Department of Mathematics and Physics, University of New Haven,
300 Boston Post Road, West Haven, CT 06516, USA**

We construct a simple Lagrangian from the scale factor of the Universe, which gives the Friedmann equations for a closed, homogeneous, and isotropic Universe. If the Universe was formed in the interior of a black hole, then the constant energy of the Universe is related to the mass of the black hole. We apply Schwinger's variational principle to this Lagrangian and show that, for a supermassive black hole, it may be the origin of the observed order of magnitude of the temperature fluctuations in the cosmic microwave background.

We consider the following Lagrangian for a homogeneous and isotropic Universe:

$$L = \dot{a}^2 + \frac{1}{3}\kappa c^2 \epsilon a^2, \quad (1)$$

where $a = a(t)$ is the scale factor as a function of the cosmic time t , a dot denotes the derivative with respect to t , $\kappa = 8\pi G/c^4$, and $\epsilon = \epsilon(a)$ is the energy density of matter in the Universe. Regarding the scale factor as a generalized coordinate, the energy of the Universe is given by

$$E = \frac{\partial L}{\partial \dot{a}} \dot{a} - L = 2\dot{a}^2 - \dot{a}^2 - \frac{1}{3}\kappa c^2 \epsilon a^2 = \dot{a}^2 - \frac{1}{3}\kappa c^2 \epsilon a^2, \quad (2)$$

and is constant because the Lagrangian does not depend explicitly on time [1]. Denoting the constant E by $-kc^2$, we obtain

$$\frac{\dot{a}^2}{c^2} + k = \frac{1}{3}\kappa \epsilon a^2. \quad (3)$$

This equation is the first Friedmann equation for a homogeneous and isotropic Universe which can be closed ($k = 1$), flat ($k = 0$), or open ($k = -1$) [2].

The Lagrange equation for the generalized coordinate a is given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{a}} - \frac{\partial L}{\partial a} = 0. \quad (4)$$

For the Lagrangian (1), it gives

$$\frac{2\ddot{a}}{c^2} = \frac{2}{3}\kappa \epsilon a + \frac{1}{3}\kappa a^2 \frac{\partial \epsilon}{\partial a}. \quad (5)$$

We consider a closed Universe. If U is the energy of the matter in the Universe, then its energy density and pressure are given by

$$\epsilon = \frac{U}{V}, \quad p = -\frac{\partial U}{\partial V}, \quad (6)$$

where $V = 2\pi^2 a^3$ is the volume of the Universe [2, 3]. Consequently, we find

$$\frac{\partial \epsilon}{\partial a} = \frac{\partial(U/V)}{\partial V} \frac{\partial V}{\partial a} = \left(V^{-1} \frac{\partial U}{\partial V} - \frac{U}{V^2} \right) \frac{\partial V}{\partial a} = -\frac{p + \epsilon}{V} \frac{\partial V}{\partial a} = -3 \frac{p + \epsilon}{a}. \quad (7)$$

Substituting (7) into (5) gives

$$\frac{2\ddot{a}}{c^2} + \frac{1}{3}\kappa \epsilon a = -\kappa p a, \quad (8)$$

*Electronic address: NPoplawski@newhaven.edu

or

$$\frac{2a\ddot{a}}{c^2} + \frac{1}{3}\kappa\epsilon a^2 = -\kappa p a^2. \quad (9)$$

Substituting (3) into (9) gives

$$\frac{\dot{a}^2 + 2a\ddot{a}}{c^2} + 1 = -\kappa p a^2. \quad (10)$$

This equation is the second Friedmann equation for a closed, homogeneous, and isotropic Universe.

To have a unit of energy, the Lagrangian (1) must be multiplied by a constant mass. This multiplication does not alter the Lagrange equation (4). We assume that the closed Universe is the interior of a black hole existing in another universe [4]. In this case, such a constant is equal to the mass of the black hole M . Accordingly, the constant energy $-kc^2$ becomes, for $k = 1$, equal to $-Mc^2$, which is the energy of the closed Universe since the energy of the black hole is equal to Mc^2 and the energy of a completely closed universe is zero [5]. The Lagrangian (1) therefore becomes

$$L = M\left(\dot{a}^2 + \frac{1}{3}\kappa c^2 \epsilon a^2\right). \quad (11)$$

To avoid a singularity in a black hole and allow its interior to form a closed universe, the general theory of relativity must be extended. The conservation law for the angular momentum in curved spacetime, consistent with the Dirac equation, requires that the antisymmetric part of the affine connection (torsion tensor) [6] is a variable in the principle of least action [7]. The simplest and most natural theory of gravity with torsion is the Einstein-Cartan (EC) theory [8]. In this theory, torsion is coupled to the spin of fermions. The multipole expansion of the conservation law for the spin tensor in EC gives a spin tensor which describes fermionic matter as a spin fluid [9]. The effective energy density and pressure of a spin fluid are given by

$$\tilde{\epsilon} = \epsilon - \alpha n_f^2, \quad \tilde{p} = p - \alpha n_f^2, \quad (12)$$

where $\alpha = \kappa(\hbar c)^2/32$ and n_f is the number density of fermions [10]. In the Friedmann equations (3) and (10), the energy density and pressure must be replaced by $\tilde{\epsilon}$ and \tilde{p} [11]. At extremely high densities in fermionic matter, approximated as a spin fluid, the negative term αn_f^2 becomes significant and k is negligible. When this term is equal in magnitude to ϵ , which occurs at a finite a , $\tilde{\epsilon}$ is zero and so is \dot{a} . Consequently, a never reaches zero and the singularity in the universe is avoided.

The coupling between the spin of fermions and torsion in EC generates gravitational repulsion at extremely high densities [13] and thus avoids the formation of a singularity in a black hole. The collapsing matter in a black hole bounces at a finite density and then expands into a new region of space on the other side of the event horizon, which may be regarded as a nonsingular, closed universe. Quantum particle production [14] caused by an extremely high curvature near a bounce creates enormous amounts of matter, produces entropy, and generates a period of exponential expansion (inflation) of this universe [15]. This scenario can thus explain inflation [16] without a scalar field and reheating. Contrary to the scalar-field models of inflation in which a scalar field causes inflation and then decays into matter, quantum particle production near a bounce creates matter and causes inflation which ends when torsion becomes weak. Depending on the particle production rate, such a universe may undergo several nonsingular bounces until it has enough matter to reach a size at which the cosmological constant starts cosmic acceleration [15, 17]. The last bounce can be regarded as the big bang (or rather the big bounce) of this universe.

In a quantum theory, Hamilton's principle of least action [1, 2] is generalized to Schwinger's variational principle [18], according to which the variation of the transition amplitude between an initial state $|\alpha_i\rangle$ and a final state $|\alpha_f\rangle$ is equal to i/\hbar times the matrix element connecting the two states of the variation δS of the action integral S :

$$\delta\langle\alpha_f|\alpha_i\rangle = \frac{i}{\hbar}\langle\alpha_f|\delta S|\alpha_i\rangle. \quad (13)$$

For any operator O in the Heisenberg picture, this principle gives

$$\delta O = -\frac{i}{\hbar}[O, \delta S], \quad (14)$$

where square brackets denote a commutator and δS is the variation of the action at the boundary of the integration domain.

We apply Schwinger's principle for the very early Universe in a black hole, in which quantum effects are important. The scale factor a is regarded as an operator in this principle, and the integration is over time. For the Lagrangian (11), the action is given by $S = \int L dt$. Accordingly, its variation is, using $\delta\dot{a} = d\delta a/dt$,

$$\delta S = \int \left(\frac{\partial L}{\partial a} \delta a + \frac{\partial L}{\partial \dot{a}} \delta \dot{a} \right) dt = \int \left(\frac{\partial L}{\partial a} - \frac{d}{dt} \frac{\partial L}{\partial \dot{a}} \right) \delta a dt + \int \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{a}} \delta a \right) dt. \quad (15)$$

The first term on the right-hand side of (15) vanishes because of the Lagrange equation (4). Consequently, we obtain

$$\delta S = \frac{\partial L}{\partial \dot{a}} \delta \dot{a} = 2M\dot{a}\delta a, \quad (16)$$

evaluated at the boundary of the time integration domain. The relation (14) for $O = a$ gives thus

$$\delta a = -\frac{i}{\hbar} [a, 2M\dot{a}\delta a], \quad (17)$$

which is satisfied if

$$[a, \dot{a}] = \frac{i\hbar}{2M}. \quad (18)$$

This equation is the commutation relation for the scale factor a and its time derivative \dot{a} . The uncertainty principle corresponding to this commutation relation is

$$\Delta a \Delta \dot{a} \geq \frac{\hbar}{4M}, \quad (19)$$

where Δ denotes the standard deviation. The uncertainty of the scale factor Δa in a homogeneous and isotropic universe causes the primordial fluctuations of the matter density.

After the big bounce, the scale factor a grows rapidly, inflation begins, and \dot{a} exceeds the speed of light c . We can thus take $\Delta \dot{a} \sim c$. At this moment, the uncertainty modes Δa exit the cosmological horizon, giving

$$\frac{\Delta a}{a} \Big|_{\text{hor}} \sim \frac{\hbar}{4Mca_{\text{hor}}}. \quad (20)$$

Also, at this moment, the scale factor is $a_{\text{hor}} \approx \sqrt{2}a_{\text{min}}$, where a_{min} is the minimum scale factor (at the big bounce). The fluctuations Δa give the spatial density perturbations (the density contrast δ) and thus the primordial fluctuations of the temperature T [19]:

$$\frac{\Delta T}{T} \Big|_{\text{hor}} \sim \delta_{\text{hor}} \sim \frac{\Delta a}{a} \Big|_{\text{hor}} \sim \frac{\hbar}{4\sqrt{2}Mca_{\text{min}}}. \quad (21)$$

After exiting the horizon, the density contrast (as long as it is small) in the long-wavelength limit grows with the scale factor according to a power law [19]. For an exponential expansion (inflation) $\delta \sim a^0$, for the radiation-dominated era $\delta \sim a^2$, and for the matter-dominated era $\delta \sim a^1$. The largest δ will be reached when the uncertainty modes enter the horizon at recombination (after the epoch of the matter-radiation equality). Accordingly, we obtain the large angular-scale fluctuations of the temperature of the cosmic microwave background (formed at recombination):

$$\frac{\Delta T}{T} \Big|_{\text{rec}} \sim \frac{\Delta T}{T} \Big|_{\text{hor}} \left(\frac{a_i}{a_{\text{hor}}} \right)^0 \left(\frac{a_{\text{eq}}}{a_i} \right)^2 \left(\frac{a_{\text{rec}}}{a_{\text{eq}}} \right)^1 \sim \frac{\hbar a_{\text{eq}} a_{\text{rec}}}{Ma_{\text{min}}^3 N^2} \sim \frac{\hbar a_0^2}{Ma_{\text{min}}^3 N^2 (1 + z_{\text{eq}})(1 + z_{\text{rec}})}, \quad (22)$$

where $a_i \sim Na_{\text{hor}}$ is the scale factor at the end of inflation, N is the expansion factor during inflation, a_{eq} is the scale factor at the matter-radiation equality, a_{rec} is the scale factor at recombination, a_0 is the present scale factor, z_{eq} is the redshift at the matter-radiation equality, and z_{rec} is the redshift at recombination. To solve the flatness and horizon problems, N must be greater than $\sim 10^{26}$ [16]. Such a number can result from torsion and particle production [15, 17].

Substituting into (22) the observed values of $\Delta T/T_{\text{rec}} \approx 2 \times 10^{-5}$, $z_{\text{eq}} \approx 3600$, $z_{\text{rec}} \approx 1100$ [19], and $a_0 \sim 10^{27}$ m [12], and assuming that the mass of the black hole is on the order of the mass of a typical supermassive black hole which exists at the center of every galaxy, $M \sim 10^{39}$ kg $\sim 10^9 M_{\odot}$, gives

$$a_{\text{min}} \sim 10^{-14} \text{ m}. \quad (23)$$

This scale factor is greater than the Cartan radius for an electron, $\sim 10^{-27}$ m [17, 20], but lesser than the minimum scale factor for the Universe with spin and torsion but without particle production near the big bounce, $\sim 10^{-5}$ m [12]. This result is natural because the Universe with particle production near the big bounce can start from a lower value of the minimum scale factor and reach the same mass. Accordingly, the observed order of magnitude of the temperature fluctuations in the cosmic microwave background may be a consequence of Schwinger's variational principle applied to the Lagrangian of the closed Universe formed in a supermassive black hole existing in another universe.

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